



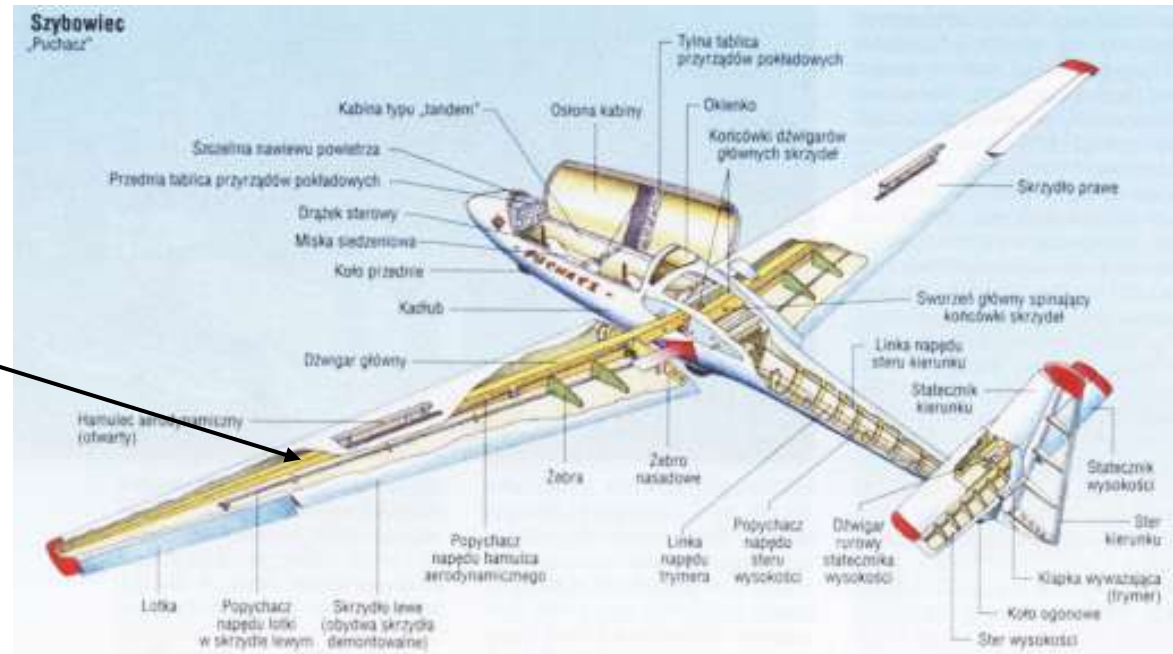
Finite element method (FEM1)

Lecture 9A. 1D beam element

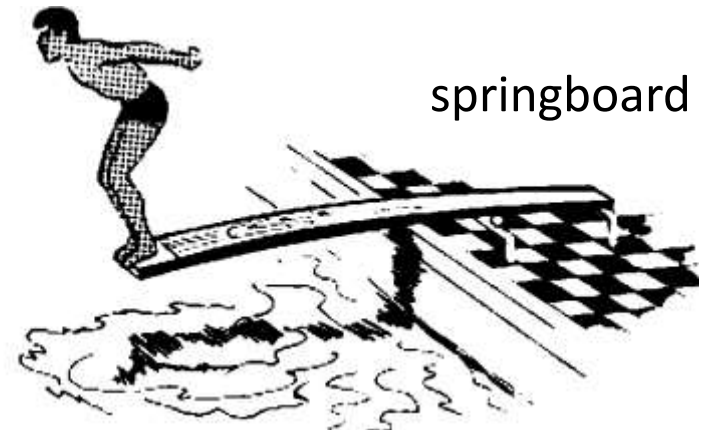
05.2025

Examples of beams

wing spar

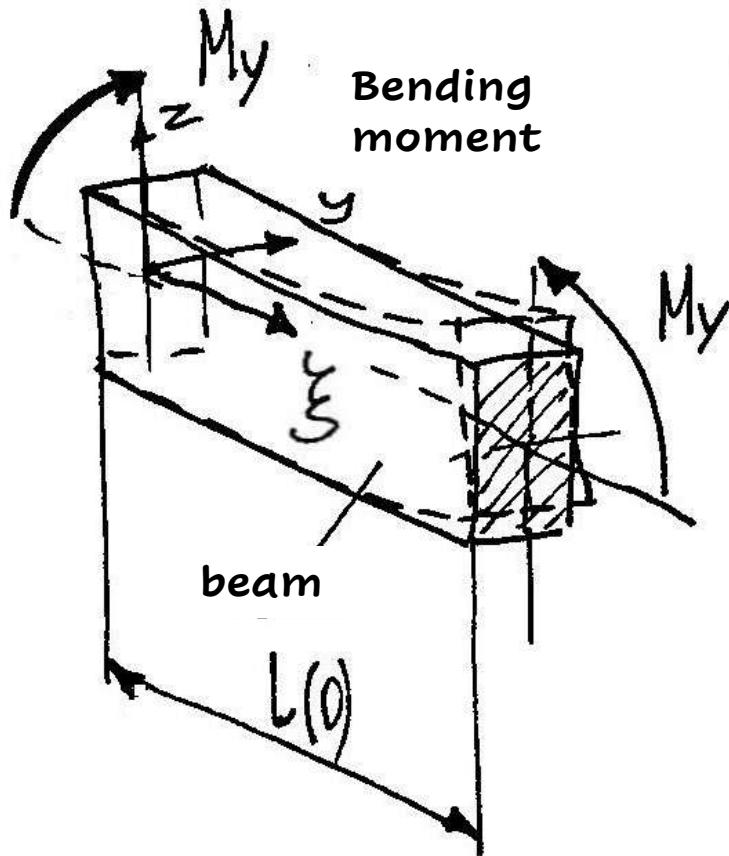


footbridge

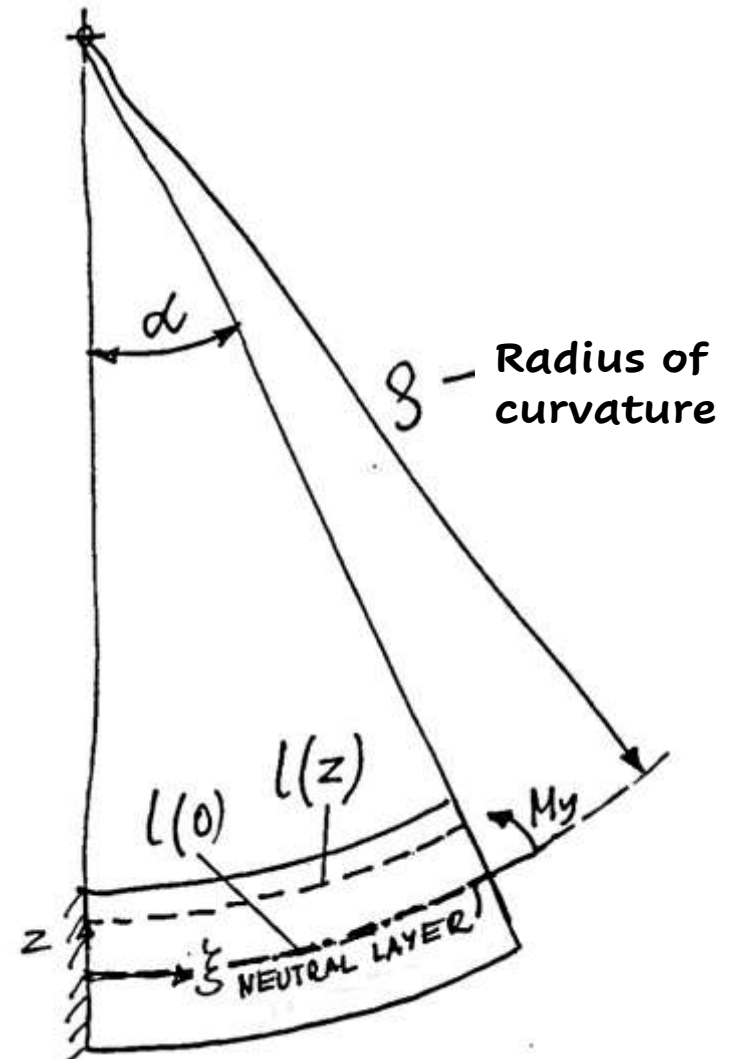


springboard

Bending without shear force (pure bending)



$$\epsilon_x(z) = \frac{l(z) - l(0)}{l(0)} = \frac{\alpha(\rho - z) - \alpha\rho}{\alpha\rho} = -\frac{z}{\rho}$$

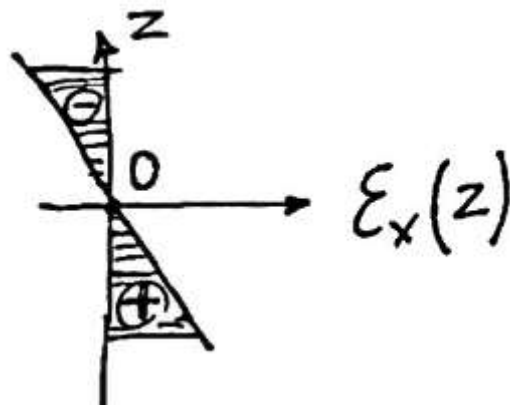


Curvature:

$$\mathcal{K} = \frac{1}{\rho} \approx \frac{d^2 w}{d\xi^2} = w''$$

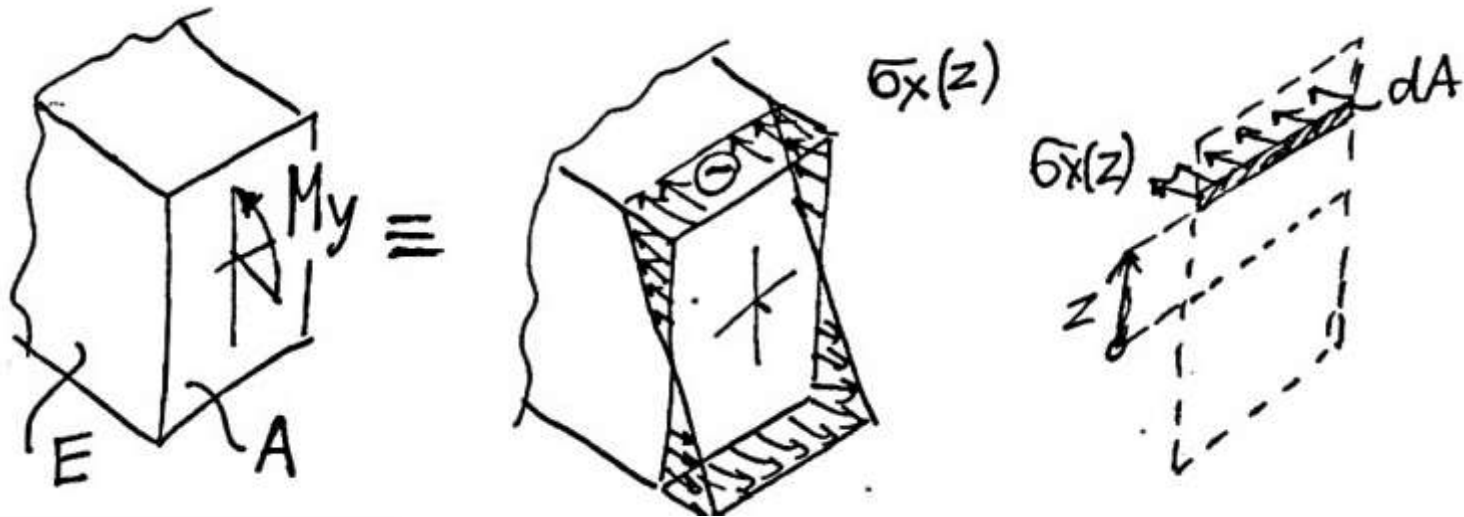
Strain:

$$\epsilon_x(z) = -z \cdot w''$$



Stress:

$$\sigma_x(z) = E \cdot \epsilon_x(z) = -E z \cdot w''$$



$$M_y = - \int_A \sigma_x(z) \cdot z \, dA = - \int_A -Ez w'' \cdot z \, dA =$$

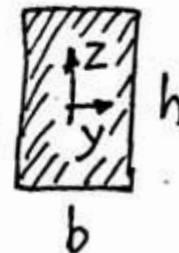
$$= E w'' \int_A z^2 \, dA \Rightarrow \boxed{M_y = E J_y w''}$$

Bending moment in a beam

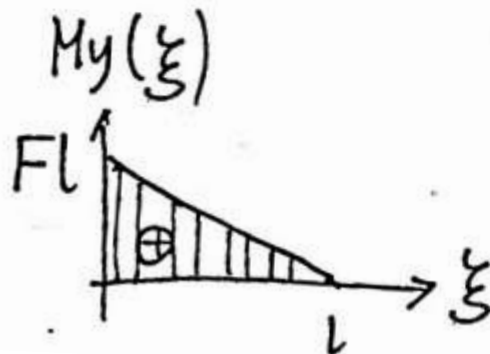
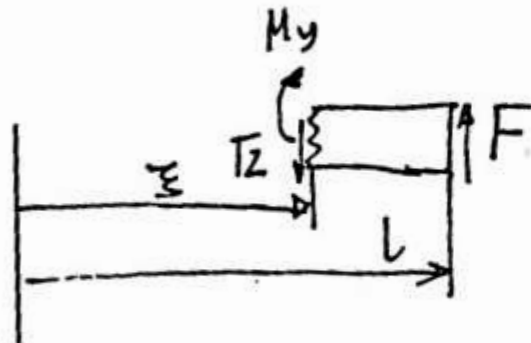
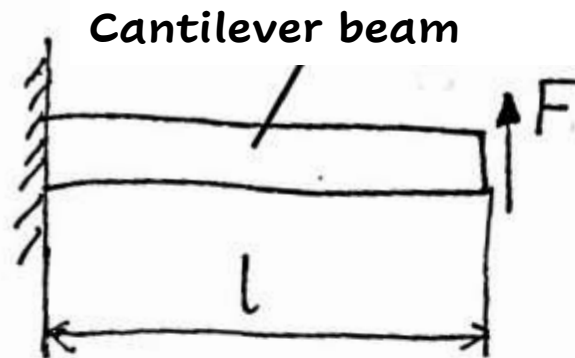
Second moment of area J_y

$$J_y = \int_A z^2 \, dA = \frac{bh^3}{12}$$

For a rectangle:



Bending with shear force(zginanie poprzeczne)



$$T_z = F$$

$$M_y - F(l - \xi) = 0$$

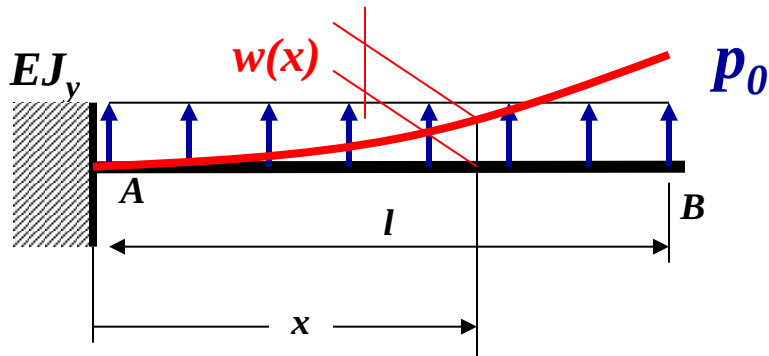
$$M_y = T_z(l - \xi)$$

$$\frac{dM_y}{d\xi} = -T_z$$

Shear force
in a beam

$$T_z = -EJ_y w'''$$

A reminder: cantilever beam - Ritz method solution



Solve a cantilever beam using the Ritz method using a given approximation function:

$$\tilde{w}(x) = a_1 + a_2 \cdot x + a_3 \cdot x^2 + a_4 \cdot x^3$$

Boundary conditions: $\tilde{w}(x=0) = 0$

$$a_1 = 0$$

$\tilde{w}'(x=0) = 0$

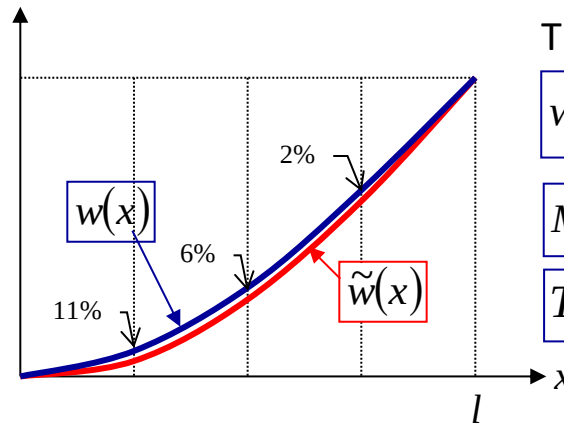
$$a_2 = 0$$

Approximate solution:

$$\tilde{w}(x) = \frac{5}{24} \frac{p_0 l^2}{EJ_y} \cdot x^2 - \frac{1}{12} \frac{p_0 l}{EJ_y} \cdot x^3$$

$$\tilde{M}_g(x) = \frac{5}{12} p_0 l^2 - \frac{1}{2} p_0 l \cdot x$$

$$\tilde{T}(x) = -\frac{1}{2} p_0 l$$

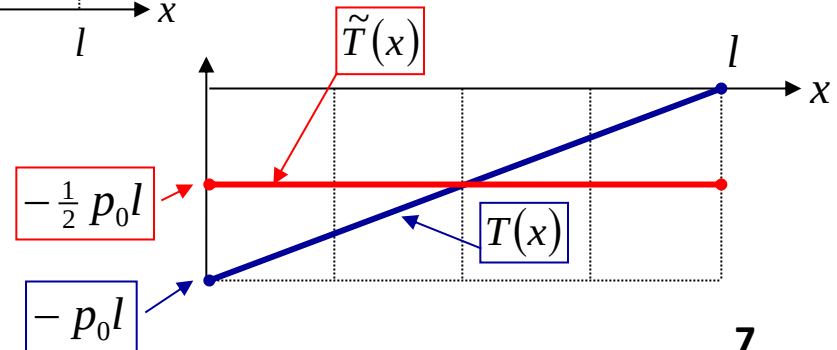
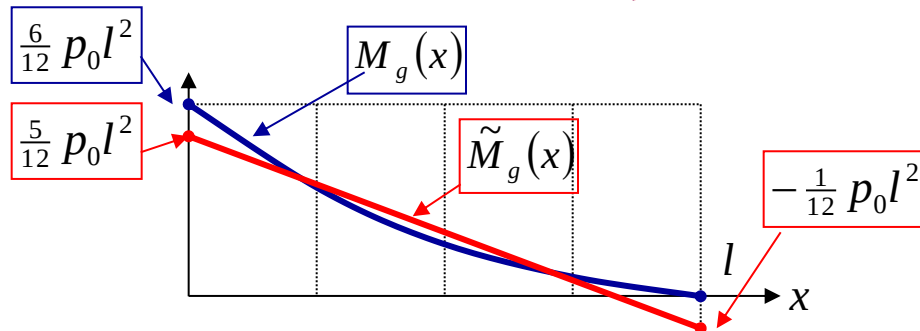


The exact solution:

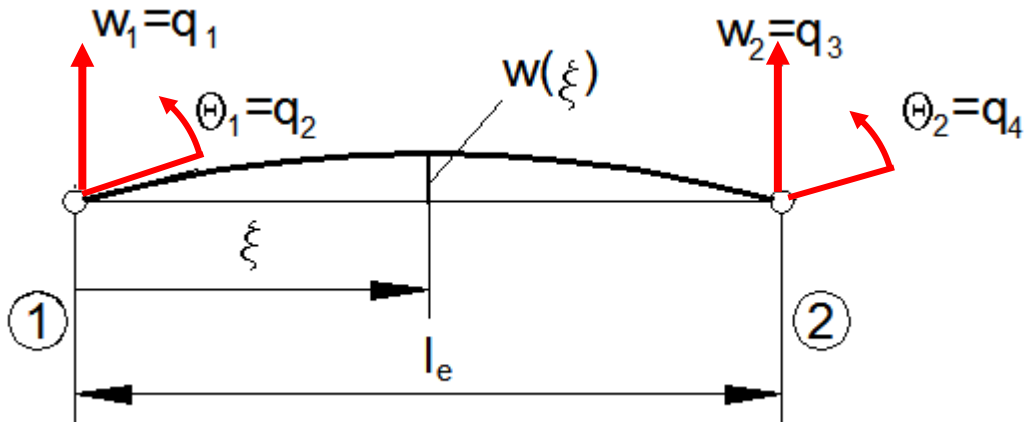
$$w(x) = \frac{6}{24} \frac{p_0 l^2}{EJ_y} \cdot x^2 - \frac{2}{12} \frac{p_0 l}{EJ_y} \cdot x^3 + \frac{1}{24} \frac{p_0}{EJ_y} \cdot x^4$$

$$M_g(x) = \frac{1}{2} p_0 (l - x)^2$$

$$T(x) = -p_0 (l - x)$$



A beam finite element (bending in one plane)



q_1, q_3 – transverse displacements at nodes
 q_2, q_4 – deflection angles at nodes
 (positive signs in counterclockwise direction)

$$n = 2 ; n_p = 2 \rightarrow n_e = n \cdot n_p = 4$$

Let us assume an approximation of the deflection function in the element:

$$w(\xi) = \alpha_1 + \alpha_2 \xi + \alpha_3 \xi^2 + \alpha_4 \xi^3$$

However, new parameters are required: $w_1, w_2, \theta_1, \theta_2$

Vector of nodal parameters:

$$\{q\}_e = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}_e = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e$$

Nodal approximation:

$$w(\xi) = \sum_{i=1}^4 N_i(\xi) q_i$$

$$w(\xi) = [N(\xi)] \{q\}_e,$$

A beam finite element - the relationship between $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and q_1, q_2, q_3, q_4

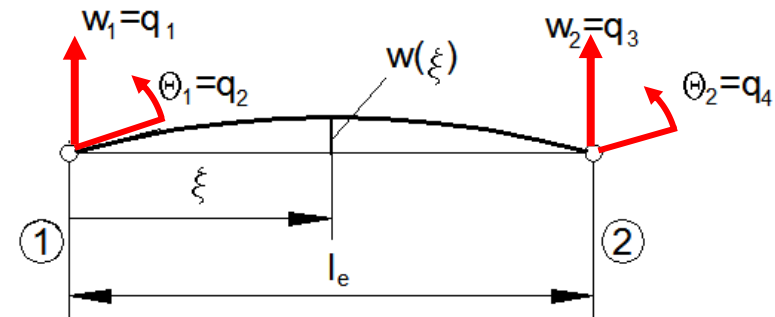
$$w(\xi) = \alpha_1 + \alpha_2 \xi + \alpha_3 \xi^2 + \alpha_4 \xi^3$$

displacement at node 1 $\rightarrow q_1 = w(0) = \alpha_1$,

deflection angle at node 1 $\rightarrow q_2 = \frac{dw}{d\xi}(0) = \alpha_2$,

displacement at node 2 $\rightarrow q_3 = w(l) = \alpha_1 + \alpha_2 l_e + \alpha_3 l_e^2 + \alpha_4 l_e^3$,

deflection angle at node 2 $\rightarrow q_4 = \frac{dw}{d\xi}(l) = \alpha_2 + 2\alpha_3 l_e + 3\alpha_4 l_e^2$.



In matrix notation:

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l_e & l_e^2 & l_e^3 \\ 0 & 1 & 2l_e & 3l_e^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \quad \rightarrow \quad \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{l_e^2} & -\frac{2}{l_e} & \frac{3}{l_e^2} & -\frac{1}{l_e} \\ \frac{2}{l_e^3} & \frac{1}{l_e} & -\frac{2}{l_e^3} & \frac{1}{l_e^2} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e$$

A beam finite element – shape functions

The approximated displacement can be represented in the form:

$$w(\xi) = \begin{bmatrix} 1, \xi, \xi^2, \xi^3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{bmatrix} N_1(\xi), N_2(\xi), N_3(\xi), N_4(\xi) \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

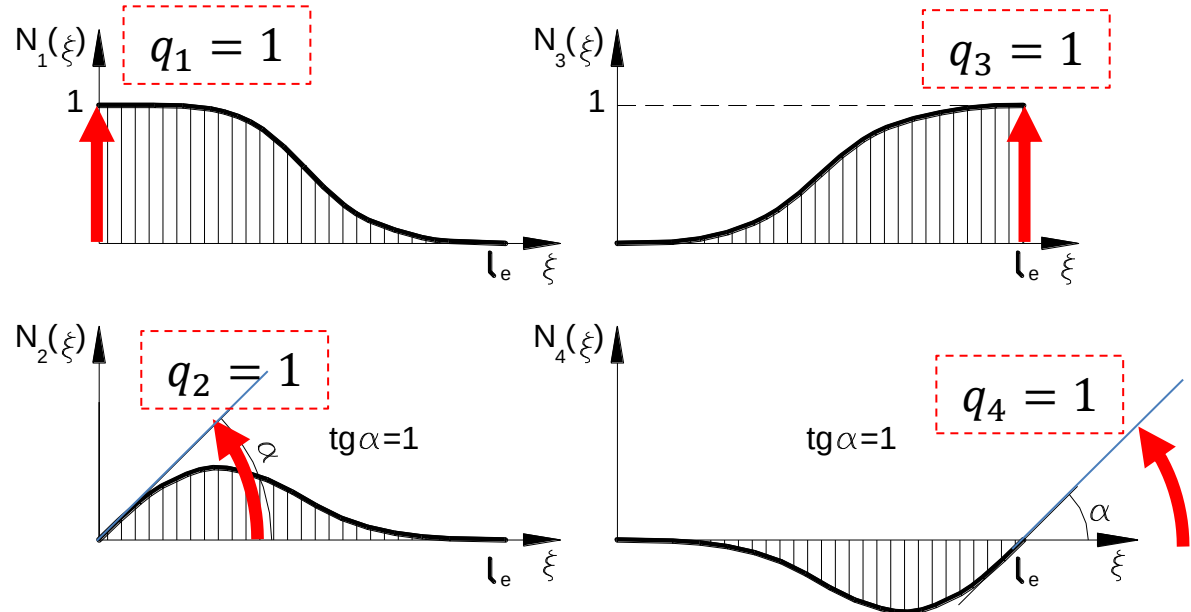
Shape functions of a beam element:

$$N_1(\xi) = 1 - 3\frac{\xi^2}{l_e^2} + 2\frac{\xi^3}{l_e^3},$$

$$N_2(\xi) = \xi - 2\frac{\xi^2}{l_e} + \frac{\xi^3}{l_e^2},$$

$$N_3(\xi) = 3\frac{\xi^2}{l_e^2} - 2\frac{\xi^3}{l_e^3},$$

$$N_4(\xi) = \frac{-\xi^2}{l_e} + \frac{\xi^3}{l_e^2}.$$



A beam finite element - shape functions and their derivatives

$$N' = \frac{dN}{d\xi}, \quad N'' = \frac{d^2 N}{d\xi^2}, \quad N''' = \frac{d^3 N}{d\xi^3}$$

For the first shape function:

$$N_1' = -\frac{6}{l^2}\xi + \frac{6}{l^3}\xi^2, \quad N_1'' = -\frac{6}{l^2} + \frac{12}{l^3}\xi, \quad N_1''' = \frac{12}{l^3}$$

For other shape functions:

$$N_2' = 1 - \frac{4}{l}\xi + \frac{3}{l^2}\xi^2, \quad N_2'' = -\frac{4}{l} + \frac{6}{l^2}\xi, \quad N_2''' = \frac{6}{l^2}$$

$$N_3' = \frac{6}{l^2}\xi - \frac{6}{l^3}\xi^2, \quad N_3'' = \frac{6}{l^2} - \frac{12}{l^3}\xi, \quad N_3''' = -\frac{12}{l^3}$$

$$N_4' = -\frac{2}{l}\xi + \frac{3}{l^2}\xi^2, \quad N_4'' = -\frac{2}{l} + \frac{6}{l^2}\xi, \quad N_4''' = \frac{6}{l^2}$$

A beam finite element – total potential energy

Deflection function and its derivatives:

$$\begin{aligned} w(\xi) &= [N(\xi)] \{q\}_e, \\ w'(\xi) &= [N'(\xi)] \{q\}_e, \\ w''(\xi) &= [N''(\xi)] \{q\}_e. \end{aligned}$$

Total potential energy of a beam of length l_e :

$$V_e = U_e - W_{ze} = \frac{EI}{2} \int_0^{l_e} (w''(\xi))^2 d\xi - \int_0^{l_e} p(\xi) w(\xi) d\xi - \sum_i P_i w_i - \sum_j M_j \vartheta_j$$

$$\begin{aligned} U_e &= \frac{EI}{2} \int_0^{l_e} w''(\xi) w''(\xi) d\xi = \frac{EI}{2} \int_0^{l_e} [q]_e \{N''\} [N''] \{q\}_e d\xi = \\ &= \frac{EI}{2} [q]_e \int_0^{l_e} \begin{bmatrix} N_1'' N_1'' & N_1'' N_2'' & N_1'' N_3'' & N_1'' N_4'' \\ N_2'' N_1'' & N_2'' N_2'' & N_2'' N_3'' & N_2'' N_4'' \\ N_3'' N_1'' & N_3'' N_2'' & N_3'' N_3'' & N_3'' N_4'' \\ N_4'' N_1'' & N_4'' N_2'' & N_4'' N_3'' & N_4'' N_4'' \end{bmatrix} d\xi \{q\}_e. \end{aligned}$$

A beam finite element – stiffness matrix

Elastic strain energy of the beam:

$$U_e = \frac{1}{2} [q]_e [k]_e \{q\}_e$$

$$[k]_e = EI \begin{bmatrix} \int_0^{l_e} N_1'' N_1'' d\xi & \int_0^{l_e} N_1'' N_2'' d\xi & \int_0^{l_e} N_1'' N_3'' d\xi & \int_0^{l_e} N_1'' N_4'' d\xi \\ \int_0^{l_e} N_2'' N_1'' d\xi & \int_0^{l_e} N_2'' N_2'' d\xi & \int_0^{l_e} N_2'' N_3'' d\xi & \int_0^{l_e} N_2'' N_4'' d\xi \\ \int_0^{l_e} N_3'' N_1'' d\xi & \int_0^{l_e} N_3'' N_2'' d\xi & \int_0^{l_e} N_3'' N_3'' d\xi & \int_0^{l_e} N_3'' N_4'' d\xi \\ \int_0^{l_e} N_4'' N_1'' d\xi & \int_0^{l_e} N_4'' N_2'' d\xi & \int_0^{l_e} N_4'' N_3'' d\xi & \int_0^{l_e} N_4'' N_4'' d\xi \end{bmatrix}$$

Stiffness matrix of a beam element:

$$[k]_e = \frac{2EI}{l_e^3} \begin{bmatrix} 6 & 3l_e & -6 & 3l_e \\ 3l_e & 2l_e^2 & -3l_e & l_e^2 \\ -6 & -3l_e & 6 & -3l_e \\ 3l_e & l_e^2 & -3l_e & 2l_e^2 \end{bmatrix}$$

A beam finite element – equivalent forces

Work of external load:

$$W_{ze}^p = \int_0^{l_e} p(\xi) w(\xi) d\xi = \int_0^{l_e} p(\xi) [N(\xi)] \{q\}_e d\xi$$

$$= \int_0^{l_e} [N_1(\xi)p(\xi), N_2(\xi)p(\xi), N_3(\xi)p(\xi), N_4(\xi)p(\xi)] \{q\}_e d\xi,$$

$$W_{ze}^p = [F_1^e, F_2^e, F_3^e, F_4^e]_e \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = [F]_e \{q\}_e$$

Nodal equivalent forces resulting from continuous load output:

$$F_i^e = \int_0^{l_e} N_i(\xi) p(\xi) d\xi$$

Example: equivalent forces resulting from a constant continuous load

Nodal forces resulting from continuous output: $F_i^e = \int_0^{l_e} N_i(\xi) p(\xi) d\xi$

For constant continuous load:

$$F_1^e = \int_0^{l_e} N_1(\xi) \cdot p_0 \cdot d\xi = \int_0^{l_e} \left(1 - \frac{3}{l_e^2} \xi^2 + \frac{2}{l_e^3} \xi^3\right) p_0 \cdot d\xi = \frac{p_0 l_e}{2}$$

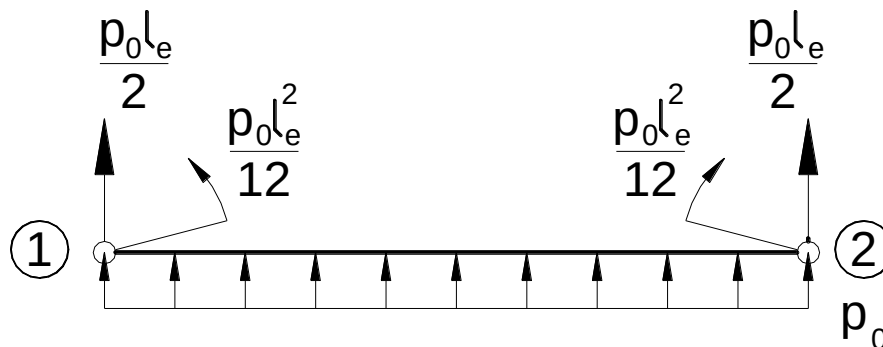
$$F_2^e = \int_0^{l_e} N_2(\xi) \cdot p_0 \cdot d\xi = \int_0^{l_e} \left(\xi - \frac{2}{l_e} \xi^2 + \frac{1}{l_e^2} \xi^3\right) p_0 \cdot d\xi = \frac{p_0 l_e^2}{12}$$

e.t.c.

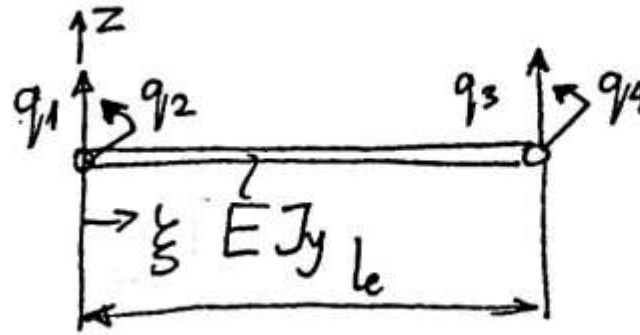
$$F_1^e = F_3^e = \frac{p_0 l_e}{2}$$

$$F_2^e = \frac{p_0 l_e^2}{12}$$

$$F_4^e = \frac{-p_0 l_e^2}{12}$$



A beam finite element – list of search functions



$$\{q\}_e = [q_1, q_2, q_3, q_4]_{1 \times 4}$$

Deflection:

$$w(\xi) = [N]_{1 \times 4} \cdot \{q\}_e - \text{Polynomial of the 3rd order}$$

Bending
moment:

$$M_y(\xi) = EJ_y w'' = EJ_y [N'']_{1 \times 4} \cdot \{q\}_e - \text{Linear function}$$

Shear
force:

$$T_z(\xi) = -EJ_y w''' = -EJ_y [N''']_{1 \times 4} \cdot \{q\}_e - \text{Constant value}$$

DOF
Solution

$$\{q\}_{N \times 1} = [K]_{N \times N}^{-1} \cdot \{F\}_{N \times 1}$$

Element
solution

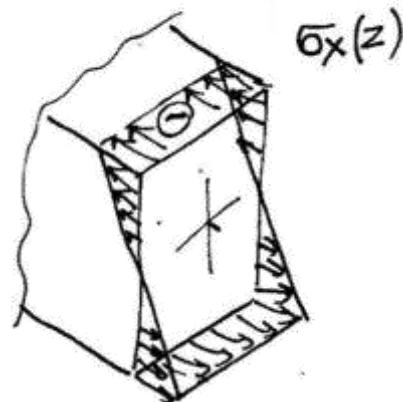
$$M_y(\xi) = E J_y \left[N''(\xi) \right]_{1 \times 4} \cdot \{q\}_{4 \times 1} e, \quad T_z(\xi) = -E J_y \left[N'''(\xi) \right]_{1 \times 4} \cdot \{q\}_{4 \times 1} e$$

$$\sigma_x(\xi, z) = -M_y(\xi) \cdot \frac{z}{J_y} = -E \left[N''(\xi) \right]_{1 \times 4} \cdot \{q\}_{4 \times 1} e \cdot z$$

$$\tau_{xz}(\xi, z) = f(z) \cdot T_z(\xi) \frac{\text{rectangle}}{b \times h} \frac{3}{2} \left(1 - \left(\frac{2z}{h} \right)^2 \right) / b h \cdot T_z(\xi)$$

$$\epsilon_x(\xi, z) = \sigma_x(\xi, z) / E = - \left[N''(\xi) \right]_{1 \times 4} \cdot \{q\}_{4 \times 1} e \cdot z$$

$$w(\xi) = \left[N(\xi) \right]_{1 \times 4} \cdot \{q\}_{4 \times 1} e$$



A beam finite element – system of equations

Total potential energy of the beam element:

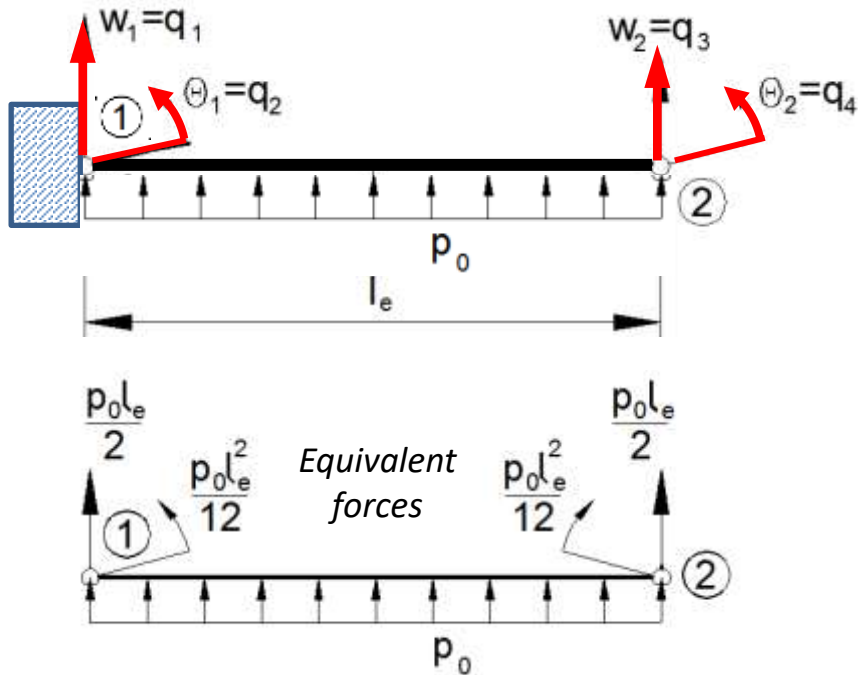
$$V_e = U_e - W_{ze} = \frac{1}{2} \underbrace{\begin{bmatrix} q \end{bmatrix}_e}_{1 \times 4} \underbrace{\begin{bmatrix} k \end{bmatrix}_e}_{4 \times 4} \underbrace{\{q\}_e}_{4 \times 1} - \underbrace{\begin{bmatrix} q \end{bmatrix}_e}_{1 \times 4} \underbrace{\{F\}_e}_{4 \times 1}$$

Condition for minimizing total potential energy:

$$\frac{\partial V_e}{\partial q_i} = 0 \quad i = 1, 2, 3, \dots, n$$

$$\boxed{\begin{bmatrix} k \end{bmatrix}_e \{q\}_e = \{F\}_e} \quad \Rightarrow \quad \frac{2EI}{l_e^3} \begin{bmatrix} 6 & 3l_e & -6 & 3l_e \\ 3l_e & 2l_e^2 & -3l_e & l_e^2 \\ -6 & -3l_e & 6 & -3l_e \\ 3l_e & l_e^2 & -3l_e & 2l_e^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}_e$$

Example: cantilever beam loaded with a uniformly distributed transverse load (*one element*)



Vector of nodal parameters:

$$\{q\}_e = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}_e = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e = \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \end{Bmatrix}_e$$

$$\frac{2EI}{l^3} \begin{bmatrix} 6 & 3l & 6 & 3l \\ 3l & 2l^2 & 3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ \frac{p_0 l}{2} \\ -\frac{p_0 l^2}{12} \end{Bmatrix}$$

$$\begin{aligned} \frac{2EI}{l^3} (6q_3 - 3lq_4) &= \frac{p_0 l}{2}, \\ \frac{2EI}{l^3} (-3lq_3 + 2l^2 q_4) &= -\frac{p_0 l^2}{12}, \end{aligned}$$



$$q_3 = \frac{1}{8} \frac{p_0 l^4}{EI}$$

$$q_4 = \frac{1}{6} \frac{p_0 l^3}{EI}$$

$$w(\xi) = \sum_{i=1}^4 N_i(\xi) q_i$$



$$w(\xi) = \left(\frac{3}{8} - \frac{1}{6} \right) \frac{p_0 l^2}{EI} \xi^2 + \left(\frac{-2}{8} + \frac{1}{6} \right) \frac{p_0 l}{EI} \xi^3 = \frac{5}{24} \frac{p_0 l^2}{EI} \xi^2 - \frac{p_0 l}{12EI} \xi^3$$

Example: cantilever beam loaded with a uniformly distributed transverse load (*one element*)

Reactions:

$$\begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ l^3 & -6 & 3l & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ \frac{p_0 l}{2} \\ -\frac{p_0 l^2}{12} \end{Bmatrix}$$

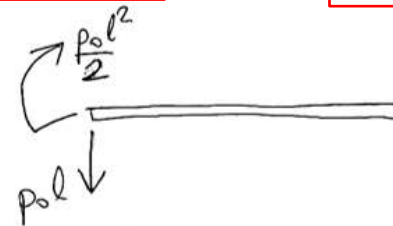
$$q_3 = \frac{1}{8} \frac{p_0 l^4}{EI} \quad q_4 = \frac{1}{6} \frac{p_0 l^3}{EI}$$



$$\begin{cases} \frac{2EI}{l^3} (-6 \cdot q_3 + 3l \cdot q_4) = R_1 + \frac{p_0 l}{2} \\ \frac{2EI}{l^3} (-3l \cdot q_3 + l^2 \cdot q_4) = R_2 + \frac{p_0 l^2}{12} \end{cases}$$

$$R_1 = -p_0 l$$

$$R_2 = -\frac{p_0 l^2}{2}$$

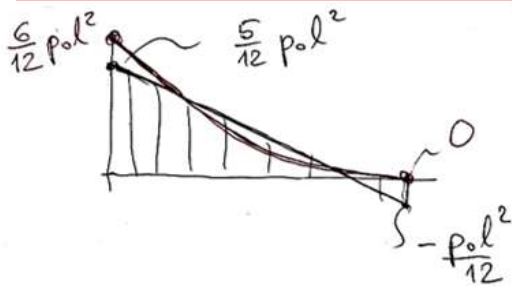


Bending moment:

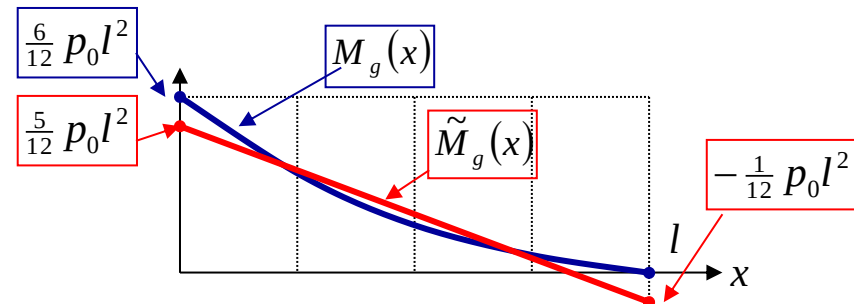
$$M_g = EJ \cdot w''(\xi) = EJ \cdot \sum_i N_i''(\xi) \cdot q_i$$

$$M_g = EJ (N_3''(\xi) \cdot q_3 + N_4''(\xi) \cdot q_4)$$

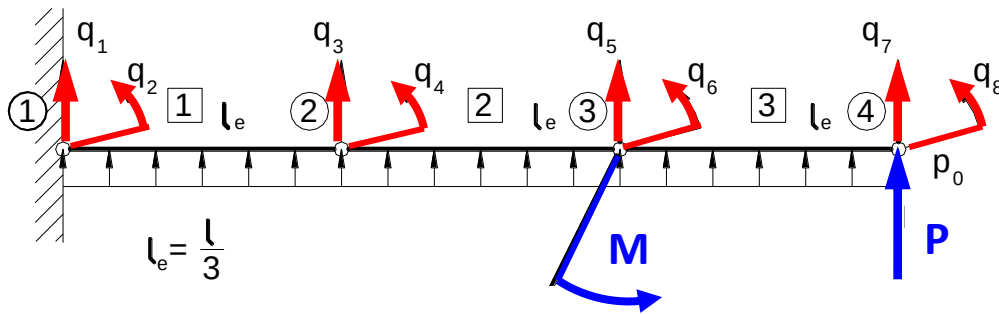
$$M_g = \frac{5}{12} p_0 l^2 - \frac{p_0 l}{2} \cdot \xi$$



As in the Ritz solution!



Example: cantilever beam loaded with a uniformly distributed transverse load, nodal loads
(*three elements*)



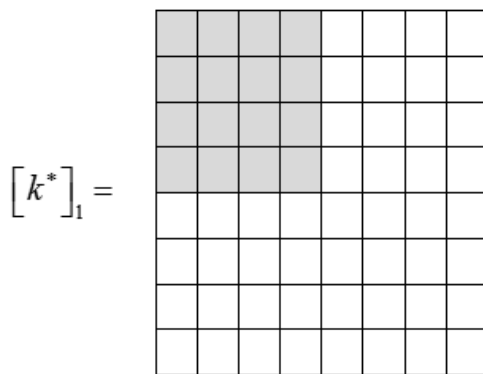
Vector of nodal parameters:

$$\{q\} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \\ w_4 \\ \theta_4 \end{Bmatrix}$$

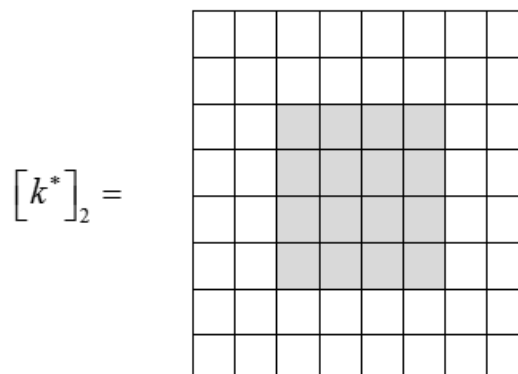
Elastic strain energy in every element:

$$U_e = \frac{1}{2} [q]_e [k]_e \{q\}_e = \frac{1}{2} [q]_{1 \times N} [k^*]_{N \times N} \{q\}_{N \times 1}$$

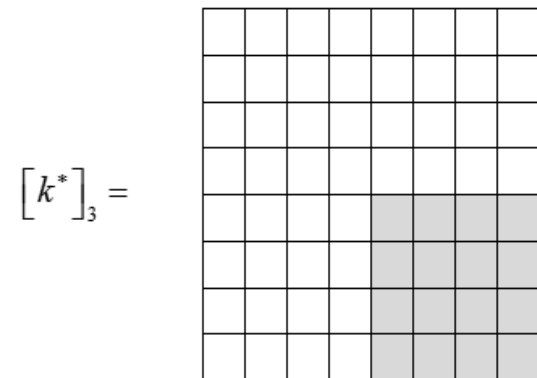
Extended Element Stiffness Matrices:



q_1, q_2, q_3, q_4



q_3, q_4, q_5, q_6



q_5, q_6, q_7, q_8

Example: cantilever beam loaded with a uniformly distributed transverse load, nodal loads
(*three elements*)

Elastic strain energy of the entire beam:

$$U = \sum_{e=1}^{LE} U_e = \frac{1}{2} [q] \left(\sum_{i=1}^{LE} [k^*]_e \right) \{q\} = \frac{1}{2} [q] [K] \{q\}$$

Total potential energy of the system:

$$V = U - W_z = \frac{1}{2} [q] [K] \{q\} - [q] \{F\}$$

The condition of minimum total potential energy of the system:

$$\frac{\partial V}{\partial q_i} = 0 \quad i = 1, 2, 3, \dots, n$$

$$[K] \{q\} = \{F\} \quad + \text{displacement boundary conditions}$$

$$M_q(\xi) = EI w''(\xi) = EI \begin{bmatrix} N_1'' & N_2'' & N_3'' & N_4'' \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e, \quad M_q(\xi) = \left[\frac{12}{l_e^3} \left(\xi - \frac{l_e}{2} \right) q_1 + \frac{6}{l_e^2} \left(\xi - \frac{2}{3} l_e \right) q_2 - \frac{12}{l_e^3} \left(\xi - \frac{l_e}{2} \right) q_3 + \frac{6}{l_e^2} \left(\xi - \frac{l_e}{3} \right) q_4 \right] EI,$$

$$T(\xi) = -EI w'''(\xi) = EI \begin{bmatrix} N_1''' & N_2''' & N_3''' & N_4''' \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e. \quad T(\xi) = - \left[\frac{12}{l_e^3} (q_1 - q_3) + \frac{6}{l_e^2} (q_2 + q_4) \right] EI.$$

Example: cantilever beam loaded with a uniformly distributed transverse load, nodal loads
(three elements)

k_{11}^1	k_{12}^1	k_{13}^1	k_{14}^1	0	0	0	0
k_{21}^1	k_{22}^1	k_{23}^1	k_{24}^1	0	0	0	0
k_{31}^1	k_{32}^1	$k_{33}^1 + k_{11}^2$	$k_{34}^1 + k_{12}^2$	k_{13}^2	k_{14}^2	0	0
k_{41}^1	k_{42}^1	$k_{43}^1 + k_{21}^2$	$k_{44}^1 + k_{22}^2$	k_{23}^2	k_{24}^2	0	0
0	0	k_{31}^2	k_{32}^2	$k_{33}^2 + k_{11}^3$	$k_{34}^2 + k_{12}^3$	k_{13}^3	k_{14}^3
0	0	k_{41}^2	k_{42}^2	$k_{43}^2 + k_{21}^3$	$k_{44}^2 + k_{22}^3$	k_{23}^3	k_{24}^3
0	0	0	0	k_{31}^3	k_{32}^3	k_{33}^3	k_{34}^3
0	0	0	0	k_{41}^3	k_{42}^3	k_{43}^3	k_{44}^3

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix}$$

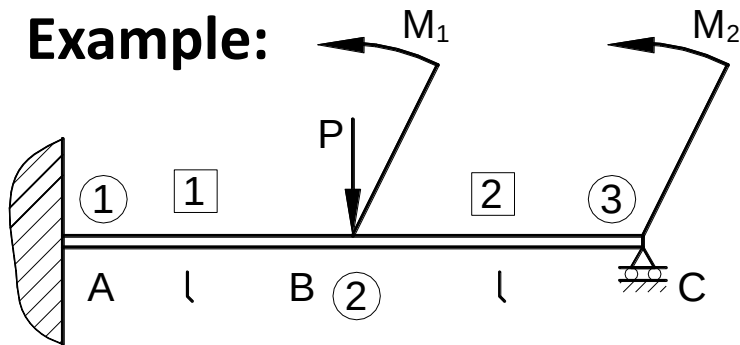
6	$3l_e$	-6	$3l_e$	0	0	0	0
$3l_e$	$2l_e^2$	$-3l_e$	l_e^2	0	0	0	0
-6	$-3l_e$	12	0	-6	$3l_e$	0	0
$3l_e$	l_e^2	0	$4l_e^2$	$-3l_e$	l_e^2	0	0
0	0	-6	$-3l_e$	12	0	-6	$3l_e$
0	0	$3l_e$	l_e^2	0	$4l_e^2$	$-3l_e$	l_e^2
0	0	0	0	-6	$-3l_e$	6	$-3l_e$
0	0	0	0	$3l_e$	l_e^2	$-3l_e$	$2l_e^2$

$$\frac{2EI}{l_e^3} \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ p_0 l_e \\ 0 \\ p_0 l_e \\ M \\ P + \frac{p_0 l_e}{2} \\ \frac{-p_0 l_e^2}{12} \end{Bmatrix}$$

Typical FEM calculations

1. Determination of the stiffness matrix of the elements $[k]_e$
2. Aggregation of the matrix of elements into the global matrix $[K]$
3. Determination of the equivalent load vector $\{F\}$
4. Introduction of boundary conditions – determination of all the searched parameters $\{q\}$
5. Determination of internal forces (moments and shear forces) and normal and shear stresses

Example:



$$[K] = \frac{2EI}{l^3}$$

6	3l	-6	3l		
3l	2l ²	-3l	l ²		
-6	-3l	6+6	-3l+3l	-6	3l
3l	l ²	-3l+3l	2l ² +2l ²	-3l	l ²
		-6	-3l	6	-3l
		3l	l ²	-3l	2l ²

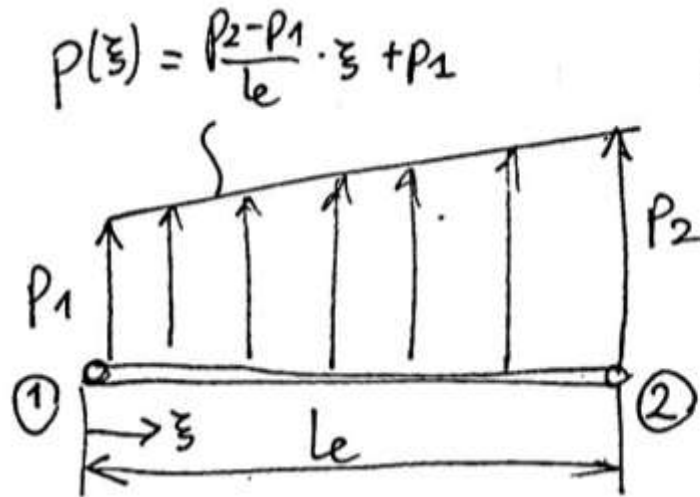
$$\{q\} = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \\ 0 \\ q_6 \end{Bmatrix}$$

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ -P \\ M_1 \\ F_5 \\ M_2 \end{Bmatrix}$$

$$\frac{2EI}{l^3} \begin{bmatrix} 12 & 0 & 3l \\ 0 & 4l^2 & l^2 \\ 3l & l^2 & 2l^2 \end{bmatrix} \begin{Bmatrix} q_3 \\ q_4 \\ q_6 \end{Bmatrix} = \begin{Bmatrix} -P \\ M_1 \\ M_2 \end{Bmatrix}$$

$$\begin{Bmatrix} q_3 \\ q_4 \\ q_6 \end{Bmatrix} = \begin{Bmatrix} w_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \frac{l}{96EI} \begin{bmatrix} 7l^2 & 3l & -12l \\ 3l & 15 & -12 \\ -12l & -12 & 48 \end{bmatrix} \begin{Bmatrix} -P \\ M_1 \\ M_2 \end{Bmatrix}$$

Example: find the components of the equivalent load for a linearly distributed transverse load



$$\{F\}_e = \{F_{1e}, F_{2e}, F_{3e}, F_{4e}\}_e$$

1) Transverse equivalent force at node 1:

$$F_{1e} = \int_0^{l_e} p(\xi) \cdot N_1(\xi) d\xi = \int_0^{l_e} \left(\frac{p_2 - p_1}{l_e} \cdot \xi + p_1 \right) \left(1 - \frac{3}{2l_e} \xi^2 + \frac{2}{l_e^3} \xi^3 \right) d\xi =$$

$$= \frac{p_1 l_e}{2} + \frac{3}{20} (p_2 - p_1) \cdot l_e$$

2) Equivalent moment at node 1:

$$\begin{aligned} F_{2e} &= \int_0^l p(\xi) \cdot N_2(\xi) d\xi = \int_0^l \left(\frac{p_2 - p_1}{l} \cdot \xi + p_1 \right) \left(\xi - \frac{2}{l} \xi^2 + \frac{1}{l^2} \xi^3 \right) d\xi = \\ &= \frac{p_1 l^2}{12} + \frac{1}{30} (p_2 - p_1) l^2 \end{aligned}$$

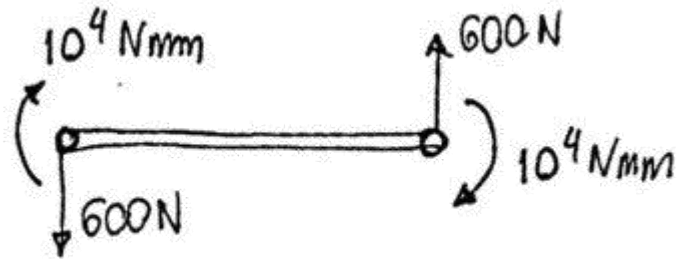
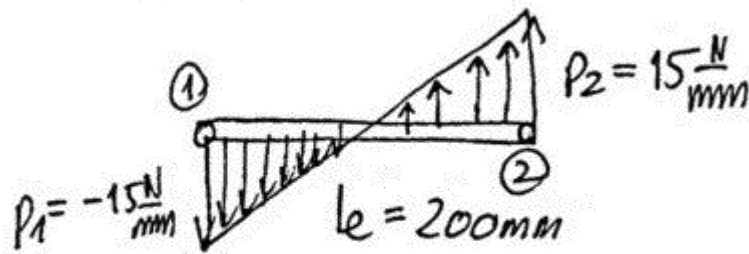
3) Transverse equivalent force at node 2:

$$\begin{aligned} F_{3e} &= \int_0^l p(\xi) N_3(\xi) d\xi = \int_0^l \left(\frac{p_2 - p_1}{l} \cdot \xi + p_1 \right) \left(\frac{3}{l^2} \xi^2 - \frac{2}{l^3} \xi^3 \right) d\xi = \\ &= \frac{p_1 l}{2} + \frac{7}{20} (p_2 - p_1) l \end{aligned}$$

4) Equivalent moment at node 2:

$$\begin{aligned} F_{4e} &= \int_0^l p(\xi) \cdot N_4(\xi) d\xi = \int_0^l \left(\frac{p_2 - p_1}{l} \cdot \xi + p_1 \right) \left(-\frac{1}{l} \xi^2 + \frac{1}{l^2} \xi^3 \right) d\xi = \\ &= -\frac{p_1 l^2}{12} - \frac{(p_2 - p_1) l^2}{20} \end{aligned}$$

Example: find the components of the equivalent load for a linearly distributed transverse load



$$F_{1e} = \frac{-15 \frac{\text{N}}{\text{mm}} \cdot 200 \text{ mm}}{2} + \frac{3}{20} \left(15 \frac{\text{N}}{\text{mm}} - (-15 \frac{\text{N}}{\text{mm}}) \right) \cdot 200 \text{ mm} = -600 \text{ N}$$

$$F_{2e} = \frac{-15 \frac{\text{N}}{\text{mm}} \cdot 200^2 \text{ mm}^2}{12} + \frac{1}{30} \left(30 \frac{\text{N}}{\text{mm}} \right) \cdot 200^2 \text{ mm}^2 = -10^4 \text{ Nmm}$$

$$F_{3e} = \frac{-15 \frac{\text{N}}{\text{mm}} \cdot 200 \text{ mm}}{2} + \frac{7}{20} \left(30 \frac{\text{N}}{\text{mm}} \right) \cdot 200 \text{ mm} = 600 \text{ N}$$

$$F_{4e} = - \left(\frac{-15 \frac{\text{N}}{\text{mm}}}{12} \right) \cdot 200^2 \text{ mm}^2 - \left(\frac{30 \frac{\text{N}}{\text{mm}}}{20} \right) \cdot 200^2 \text{ mm}^2 = -10^4 \text{ Nmm}$$